On the Stability of the $O(N)$-Invariant and the Cubic-Invariant 3-Dimensional N-Component Renormalization Group Fixed Points in the Hierarchical Approximation

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Abstract We compute renormalization group fixed points and their spectrum in an ultralocal approximation. We study a case of two competing non-trivial fixed points for a three-dimensional real N-component field: the $O(N)$-invariant fixed point vs. the cubic-invariant fixed point. We compute the critical value N_c of the cubic ϕ^4-perturbation at the $O(N)$-fixed point. The $O(N)$ fixed point is stable under a cubic ϕ^4-perturbation below N_c, above N_c it is unstable. The critical value comes out as $2.219435 < N_c < 2.219436$ in the ultralocal approximation. We also compute the critical value of N at the cubic invariant fixed point. Within the accuracy of our computations, the two values coincide.

1 Introduction

Spin systems with an N-component real field variable, governed by a Landau-Ginzburg type Hamiltonian, are of central importance both in Euclidean quantum field theory and classical statistical mechanics. The basic model is one with a global $O(N)$ symmetry, the invariance under a simultaneous rotation of all the spins. A prototypical Hamiltonial for such a model, say on a lattice $\Lambda \subset \mathbb{Z}^D$, is

$$H = \sum_{x \in \Lambda} \left\{ \frac{1}{2} \sum_{a=1}^{N} \left(\sum_{\mu=1}^{D} \partial_\mu \phi_a(x)^2 + \phi_a(x)^2 \right) + \lambda \left(\sum_{a=1}^{N} \phi_a(x)^2 \right)^2 \right\}. \tag{1}$$

It is important to study the influence of perturbations which explicitly break this symmetry to a smaller subgroup. For instance, in cubical crystals, one expects the spin interaction to react to the lattice structure. This suggests additional terms in the Hamiltonian which are not rotation symmetric, but invariant under the cubic group. The cubic group is composed of the permutations and reflections of the N components of the field. Such a cubic invariant term is

$$\mu \sum_{x \in \Lambda} \sum_{a=1}^{N} \phi_a(x)^4. \tag{2}$$

When (1) is augmented by (2), a competition of renormalization group fixed points sets in, about which of them determines the long distance behaviour of the model.

According to folklore, the $O(N)$ fixed point becomes unstable above a certain threshold value N_c. When this happens, the cubical fixed point dominates the long distance behaviour. The value of N_c is still being debated. Work on the ε-expansion suggests that $N_c > 3$. This has been challenged by the studies [12, 15, 7, 8, 9], which suggest that $N_c < 3$. Recent Monte Carlo work [3] by Caselle and Hasenbusch again indicates that N_c should be very close to 3. Within their precision, this result is compatible with that of Kleinert and collaborators [7, 8, 9].

To our knowledge, the problem has not yet been looked at in the framework of the block spin renormalization group [16]. To study the domains of attractions in a model with several competing fixed is a fundamental and challenging problem as it underlies Wilson's explanation of universality. A

notorious trouble in this business is that these Hamiltonians in general depend on an infinite number of interaction terms. To control flows with a large number of couplings is a very difficult task. Furthermore, when symmetries are reduced, the number of couplings tends to proliferate dramatically.

This paper contains a study of both the $O(N)$-invariant fixed point and the cubic fixed point in the framework of the hierarchical or ultralocal approximation to Wilson's renormalization group. Here we restrict our attention to the case of $D = 3$ dimensions. In the hierarchical approximation, the effective Hamiltonians are restricted to a (non-standard) kinetic term plus local interactions. The local interactions are given by a potential, which is a function of N variables.

2 Hierarchical Renormalization Group

The hierarchical renormalization group is a block spin renormalization group for so called hierarchical spin models. They are spin models with a non-translation invariant kinetic term designed to make the renormalization group local. The hierarchical model and the full model belong to different universality classes. Nevertheless they are related. The hierarchical renormalization group is an ultralocal approximation to a lattice block spin renormalization group. See [10, 13] and references therein. Furthermore, the hierarchical model is presumably a zeroth approximation to the full model in a lattice derivative expansion. Hierarchical results are to our experience reasonable approximations. However concerning their predictions about full models, high precision cannot be expected. For instance, ν comes out as 0.649 at the $N = 1$ Ising fixed point, as compared to 0.63 from ε-expansion and Monte-Carlo methods. Hierarchical models are known to be extremely valuable in gaining qualitative information about the model under investigation and to prepare the ground for further renormalization group studies, where non-local interactions are included.

The hierarchical renormalization group for models with an N-component real scalar field is a theory of the non-linear integral transformation

$$R_{\alpha,\beta,\gamma}(Z)(\psi) = \int d\mu_\gamma(\zeta) \, Z(\beta\psi + \zeta)^\alpha, \tag{3}$$

where α, β, and γ are real parameters, and where $d\mu_\gamma(\zeta)$ is the Gaussian measure on $\mathbb{R}^N$ with mean zero and covariance γ (times the unit matrix).

Recall that its Fourier transform is

$$\int \mathrm{d}\mu_\gamma(\zeta)\, \mathrm{e}^{\mathrm{i}\zeta j} \;=\; \mathrm{e}^{-\frac{\gamma}{2}j^2}. \tag{4}$$

We consider the renormalization group in the so called high temperature picture. See [13] and references therein. Our parameter values in (3) are

$$\alpha \;=\; 2, \qquad \beta \;=\; 2^{-\frac{2+D}{2D}}, \qquad \gamma \;=\; 1. \tag{5}$$

(You may take this as the definition of the high temperature picture.) Here D is the dimension of the model. We restrict our attention to the most interesting case, when $D = 3$.

The transformation (3) is the composition of three steps: taking the square, Gaussian convolution, and rescaling. The numerical computation of (3) will be decomposed into these steps.

3 Algebraic Formulation

We use the techniques described in [13, 6]. Consider first the $O(N)$-invariant case. A general even $O(N)$-invariant Boltzmann factor can be written as a sum

$$Z(\phi) \;=\; \sum_{n=0}^{\infty} (\phi^2)^n \, Z_n, \tag{6}$$

and is parametrized by real coefficients Z_n. The transformation (3) becomes the following non-linear transformation in terms of the coordinates (Z_n):

$$R(Z)_n \;=\; \beta^{2n} \sum_{m=n}^{\infty} G_{n,m}(N,\gamma) \sum_{l=0}^{m} Z_l\, Z_{m-l}. \tag{7}$$

The coefficients $G_{n,m}(N,\gamma)$ are defined as follows. (We call them structure coefficients.) Let $P_m(\phi) = (\phi^2)^m$. The Gaussian convolution of it is a linear combination

$$\int \mathrm{d}\mu_\gamma(\zeta)\, P_m(\psi + \zeta) \;=\; \sum_{n=0}^{m} P_n(\psi)\, G_{n,m}(N,\gamma), \tag{8}$$

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with coefficients (essentially the number of contractions)

$$G_{n,m}(N, \gamma) = \gamma^{m-n} \prod_{l=1}^{m-n} \frac{(N + 2(n + l - 1))(n + l)}{l}. \tag{9}$$

It is convenient to define $G_{n,m}(N, \gamma) = 0$ for $n > m$.

Analytical (and numerical) experience suggests a different normalization, namely $Z_n = X_n \rho^n / \sqrt{(2n)!}$ with a suitable constant ρ. For notational simplicity, we prefer to display the normalization (6).

For practical computations, one has to truncate the transformation (7) to a finite number K of non-zero coefficients. For $n \leq K$, the resulting transformation is

$$R_K(Z)_n = \beta^{2n} \sum_{m=n}^{K} G_{n,m}(N, \gamma) \sum_{l=0}^{m} Z_l Z_{m-l}. \tag{10}$$

In the high temperature picture, this truncation scheme is known to converge as $K \to \infty$. See [11] for a detailed analysis of the one component case.

Renormalization group fixed points are approximated as stationary flows of the truncated transformation

$$R_K(Z^*)_n = Z_n^*. \tag{11}$$

The most interesting datum of a renormalization group fixed point is its spectrum, from which one learns the behavior of the linearized flow in its vicinity. The linearized flows around a fixed point is given by

$$DR_K(Y)_n = 2\beta^{2n} \sum_{m=n}^{K} G_{n,m}(N, \gamma) \sum_{l=0}^{m} Z_l^* Y_{m-l}. \tag{12}$$

The spectrum of Z^* is the set of eigenvalues $\lambda^{(i)}$ of (12). The eigenvalues again are directly related to the critical exponents $\sigma^{(i)}$. See [16]. In our model, the relation is $\lambda^{(i)} = \alpha^{\sigma^{(i)}/D}$. (A peculiarity of hierarchical models is that the critical exponents are L-dependent.) The eigenvalue problem reads

$$DR_K(Y^{(i)})_n = \lambda^{(i)} Y_n^{(i)}. \tag{13}$$

The linearized renormalization group transformation (7) can be brought to a manifestly symmetric form in the no-truncation limit, and is thus diagonalizable. In practice, (13) turns out to be a very reliable way to determine the spectrum of renormalization group fixed points.

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4 Numerical Results: $O(N)$ Fixed Point

We have searched for fixed points of the system of algebraic equations (10) with a Newton algorithm. The program was written in C++ using the data type `long double` representation for real numbers. In order to check roundoff errors, we compared the `long double` with the (simple) `double` representation and found no significant deviations.

$L_{\max}$	$\lambda^{(1)}$	$\lambda^{(2)}$	$\lambda^{(3)}$	$\lambda^{(4)}$
10	1.33	0.73	0.31	0.10
12	1.36	0.79	0.38	0.15
14	1.37	0.83	0.438	0.19
16	1.384	0.849	0.469	0.22
18	1.3854	0.854	0.4846	0.25
20	1.3856	0.8560	0.48997	0.262
22	1.38573	0.8562	0.49143	0.267
24	1.385742	0.85633	0.49175	0.268
26	1.3857434	0.8563400	0.491812	0.2691
28	1.38574348	0.8563408	0.4918214	0.26923
30	1.385743489	0.85634089	0.49182258	0.269247
32	1.38574349013	0.8563409057	0.491822725	0.2692491
34	1.385743490193	0.85634090651	0.491822739	0.26924937
36	1.3857434901972	0.856340906576	0.4918227412	0.2692493996

Table 1: Spectrum of $O(2)_3$ fixed point

Table 1 shows the effect of truncation on the first few eigenvalues of the non-trivial $O(N)$ fixed point with $N = 2$ components in $D = 3$ dimensions. (A trivial "volume eigenvalue" $\lambda^{(0)} = 2$ has been omitted.) We see a rapid increase of accuracy with the number of couplings. To get the same absolute precision, the smaller the eigenvalue of interest is, the bigger the number of couplings has to be chosen. This will be important in the following.

The first eigenvalue $\lambda^{(1)}$ belongs to the critical exponent $\nu = 1/\sigma^{(1)}$, or $\nu = \log(\alpha)/D \log(\lambda)$. The numerical value of ν is thus

$$\nu = 0.7082249. \tag{14}$$

With twenty couplings, the first eigenvalue comes out with an accuracy of more than three digits, with thirty couplings of more than nine digits.
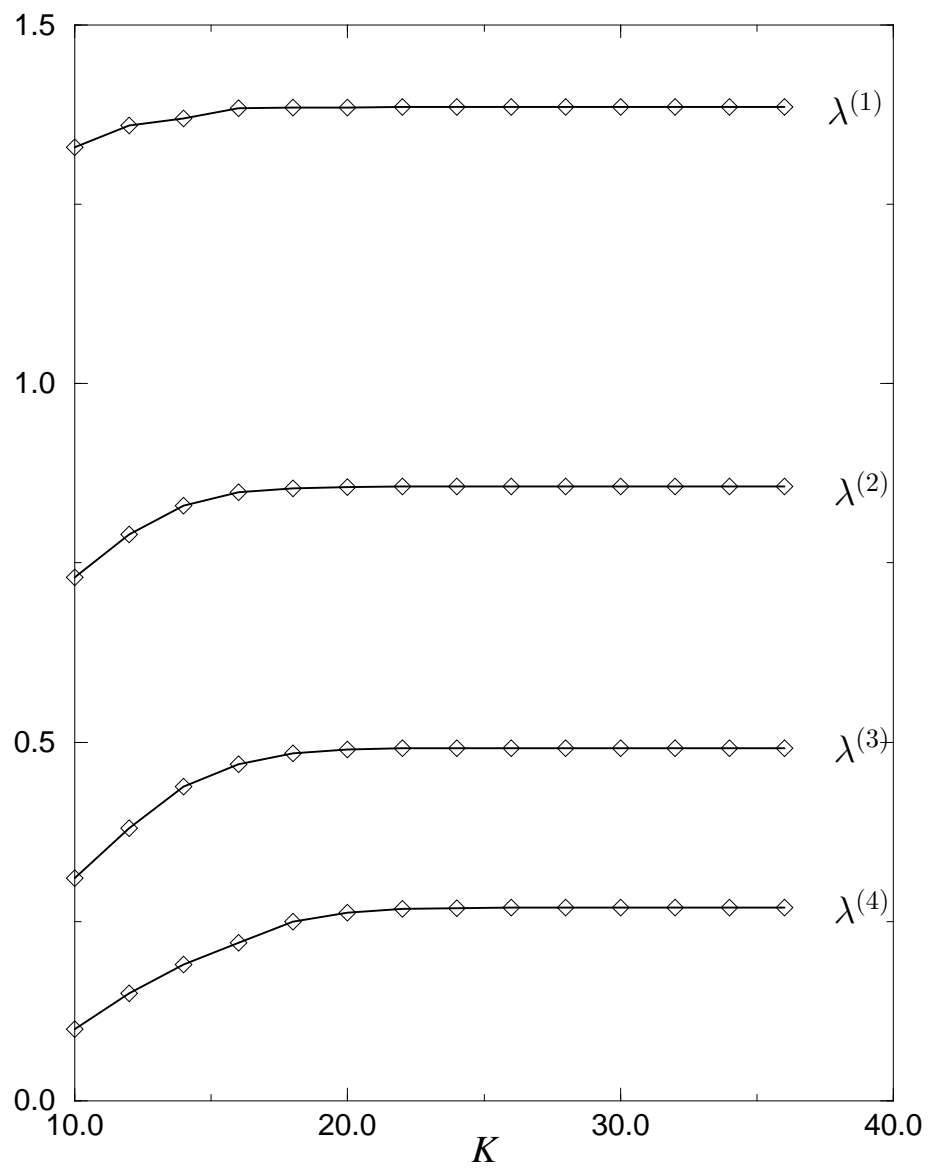
Figure 1: Spectrum of $O(2)_3$ fixed point as function of truncation order K

N	$\lambda^{(1)}$	$\lambda^{(2)}$	$\lambda^{(3)}$	$\lambda^{(4)}$
-2.2	1.59824229	0.848707935	0.435463969	0.213609627
-2.0	1.58740105	0.849947302	0.437978578	0.215734559
-1.4	1.55436959	0.853424303	0.445855470	0.222533463
-0.6	1.50993452	0.857177559	0.457002524	0.232579287
-0.2	1.48805474	0.858501342	0.462742151	0.237983202
0.2	1.466769838	0.859352758	0.468493388	0.243587863
0.6	1.446379983	0.859668728	0.474160525	0.249329758
1.0	1.427172478	0.859411649	0.479637300	0.255127880
1.4	1.409390602	0.858578978	0.484816590	0.260888802
1.6	1.401091348	0.857956757	0.487263833	0.263724669
2.0	1.385743489	0.856340897	0.491822589	0.269247586
2.6	1.365884074	0.853161847	0.497720103	0.277002053
3.0	1.354668292	0.850694212	0.500989433	0.281718313
3.2	1.349623859	0.849396273	0.502426577	0.283920286

Table 2: Spectrum of $O(N)_3$ fixed point at $K = 30$

A computer assisted proof for the one component case was developed by [11]. From it, one gets two sided bounds on the critical exponents with arbitrary precision. It confirms the accuracy of our calculation by a comparison at $N = 1$.

Table 2 shows the spectrum of the non-trivial three dimensional $O(N)$-fixed point as a function of the number of components N at truncation order $K = 30$.

Analytic continuation in N is naturally possible for the system of algebraic equations (10), since the N-dependence is encoded entirely in the structure constants (8). These depend polynomially on N. To maintain the same precision as N increases, one has to increase K. Up to $N = 3$, the truncation order $K = 30$ suffices.

We rediscover among other things the well known result that the theory becomes trivial at $N = -2$, i. e. , $\nu = 0.5$.
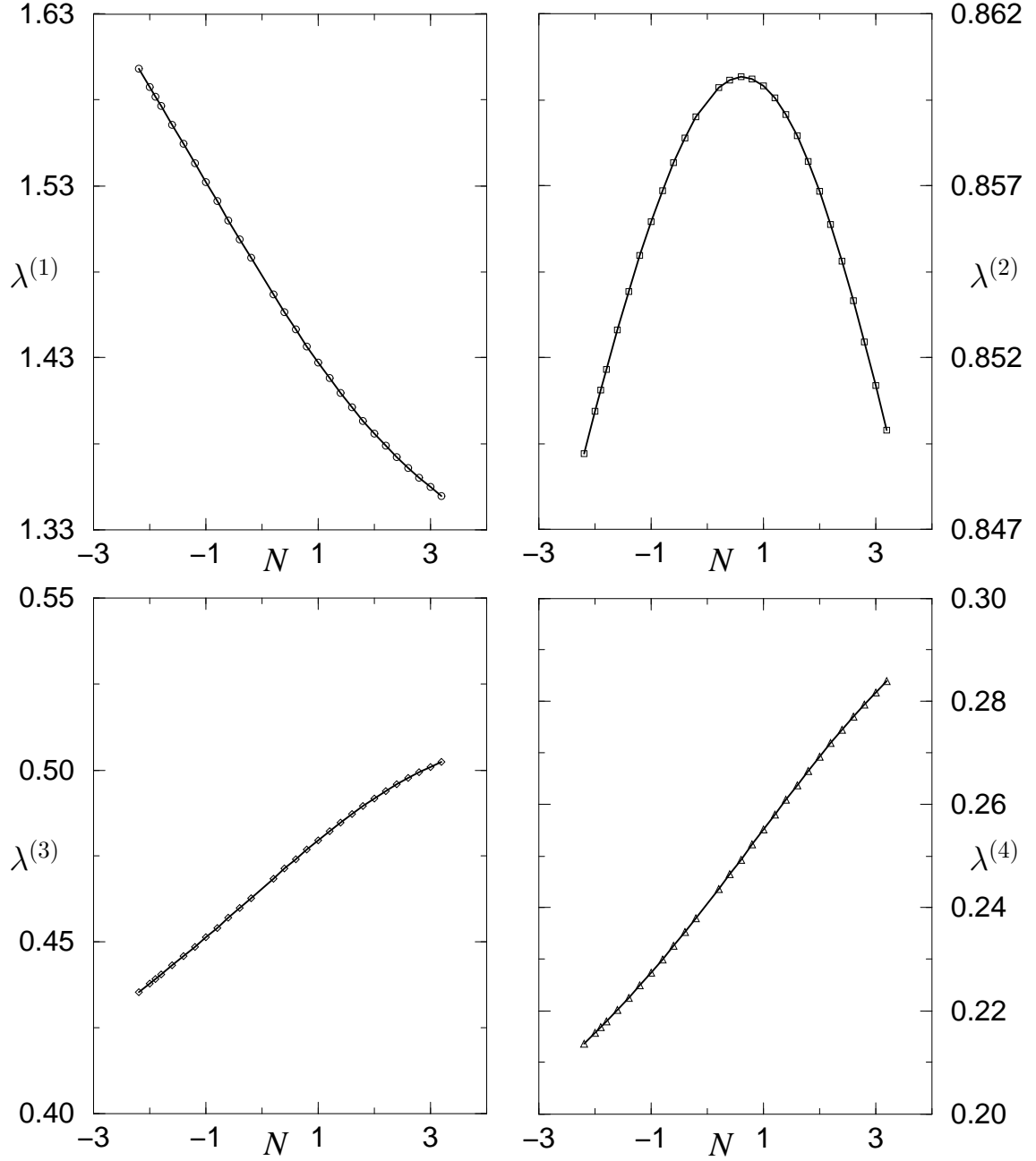
Figure 2: Spectrum of $O(N)_3$ fixed point at $K = 30$

5 Cubic Invariance

Cubic symmetry is the following. Consider an N-component model. (The local spin takes values in $\mathbb{R}^N$.) The cubic symmetry is the finite group of transformations, consisting of rotations and reflections, which leaves invariant the cube $[-1,1]^N$.

We will restrict our attention to even Boltzmann factors, with the property that

$$Z((-1)^{\sigma_1}\phi_1, \ldots, (-1)^{\sigma_N}\phi_N) \; = \; Z(\phi_1, \ldots, \phi_N). \tag{15}$$

Such Boltzmann factors are functions of $\phi_1^2, \ldots, \phi_N^2$. To be cubic invariant, they have to be in addition symmetric functions of $\phi_1^2, \ldots, \phi_N^2$. An even cubic invariant Boltzmann factor is thus a function (15), which in addition satisfies

$$Z(\phi_{\pi(1)}, \ldots, \phi_{\pi(N)}) \; = \; Z(\phi_1, \ldots, \phi_N) \tag{16}$$

for all permutations $\pi \in \mathfrak{S}_N$. We begin with the simplest case, the study of cubic perturbations of the $O(N)$-invariant fixed point.

5.1 Cubic Perturbations of the $O(N)$ Fixed Point

We can extend the polynomial basis $P_n(\phi)$ of $O(N)$-invariants as follows. Let $\underline{n} = (n_o, n_c) \in \mathbb{N} \times \{0, 1\}$ and define

$$P_{\underline{n}}(\phi) \; = \; \left(\sum_{a=1}^N \phi_a^2 \right)^{n_o} \left(\sum_{a=1}^N \phi_a^4 \right)^{n_c}. \tag{17}$$

These symmetric functions do not close under multiplication. They do however close under Gaussian convolution. They satisfy

$$\int \mathrm{d}\mu(\zeta) \, P_{\underline{m}}(\psi + \zeta) \; = \; \sum_{\underline{n}} P_{\underline{n}}(\phi) \, G_{\underline{n},\underline{m}}(N, \gamma), \tag{18}$$

with structure coefficients

$$\begin{aligned}
G_{(n,0),(m,1)}(N,\gamma) \; &= \; N \; G_{0,2}(1,\gamma) \; G_{n,m}(N+4,\gamma) \\
&\quad + G_{1,2}(1,\gamma) \; G_{n-1,m}(N+6,\gamma), \tag{19} \\
G_{(n,1),(m,1)}(N,\gamma) \; &= \; G_{2,2}(1,\gamma) \; G_{n,m}(N+8,\gamma), \tag{20} \\
G_{(n,0),(m,0)}(N,\gamma) \; &= \; G_{n,m}(N,\gamma), \tag{21}
\end{aligned}$$

where $G_{n,m}(N,\gamma)$ is as in (9). The linearized renormalization group at the $O(N)$ fixed point with cubic ϕ^4-perturbations complicates to

$$DR_K(Y)_{(n_o,n_c)} =$$

$$2\,\beta^{2n_o+4n_c} \sum_{m_o=n_o}^{K} \sum_{m_c=0}^{1} G_{(n_o,n_c),(m_o,m_c)}(N,\gamma) \sum_{l_o=0}^{m_o} Z^*_{(l_o,0)}\, Y_{(m_o-l_o,m_c)}. \qquad (22)$$

The $O(N)$-invariants, defined by $n_c = 0$, form an invariant subspace. The cubic invariant eigenvectors generally have non-vanishing $O(N)$-components. The cubic eigenvalues will be denoted by $\kappa^{(i)}$. We order them according to their degree of relevance.

N	$\lambda^{(1)}$	$\kappa^{(1)}$	$\lambda^{(2)}$	$\kappa^{(3)}$	$\lambda^{(3)}$
1.80	1.39321	0.98503	0.85721	0.54062	0.48960
1.90	1.38942	0.98865	0.85679	0.54359	0.49072
2.00	1.38574	0.99224	0.85634	0.54656	0.49182
2.10	1.38217	0.99580	0.85588	0.54950	0.49288
2.15	1.38042	0.99757	0.85562	0.55097	0.49340
2.20	1.37870	0.99932	0.85537	0.55243	0.49391
2.25	1.37701	1.00107	0.85511	0.55389	0.49442
2.30	1.37534	1.00280	0.85484	0.55534	0.49491
2.40	1.37209	1.00624	0.85430	0.55823	0.49588
2.50	1.36893	1.00965	0.85374	0.56109	0.49681

Table 3: Cubic spectrum of $O(N)_3$ fixed point at $K = 30$

Table 3 shows the $O(N)$-invariant and the cubic invariant spectrum at the $O(N)$-invariant non-trivial fixed point in three dimension as a function of the number of components N. The leading cubic eigenvector is – in ε-expansion a deformation of – a cubic ϕ^4-vertex. Here we restrict our attention to eigenvectors of the ϕ^4-type times powers of ϕ^2. (The subleading eigenvalue $\kappa^{(2)}$ belonging to the cubic ϕ^6-interaction is not displayed here.)

The largest cubic eigenvector, $\kappa^{(1)}$, becomes one at a critical value N_c of N. We learn from Table 3 that $2.20 < N_c < 2.25$. A closer look at the vicinity of N_c yields the following. Table (4) shows that the value of N_c is located inbetween

$$2.219435 < N_c < 2.219436. \qquad (23)$$

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(Numerical errors are negligible.) We learn furthermore the important lesson that the N-dependence of $\kappa^{(1)}$ is rather weak between the two and three component models.

N	K	$\kappa^{(1)}$
2.219	30	0.99998479312540
	40	0.99998479665524
2.2194	30	0.99999876426610
	40	0.99999876780069
2.21943	30	0.99999981395124
	40	0.99999981748619
2.219435	30	0.99999998889842
	40	0.99999999243344
	50	0.99999999243345
2.219536	30	1.00000002222169
	40	1.00000002575671
	50	1.00000002575672
2.21944	30	1.00000016384551
	40	1.00000016738059
2.2195	30	1.00000226320303
	40	1.00000226673882
2.220	30	1.00001973231681
	40	1.00001973585855

Table 4: N_c at the $O(N)$ fixed point

6 Cubic Invariant Fixed Point

Besides the $O(N)$-invariant fixed point, we find a cubic invariant fixed point. Again we restrict our attention to the case of three dimensions. The cubic perturbations (17) have to be enlarged to a generating system of cubic invariant polynomials. We have investigated several possibilities.

6.1 Lifted Representation

The first possibility uses an over complete system. Let $\underline{n} = (n_1, n_2, n_3, \dots) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} \times \cdots$. Define

$$P_{\underline{n}}(\phi) = \left(\sum_{a=1}^{N} \phi_a^2 \right)^{n_1} \left(\sum_{a=1}^{N} \phi_a^4 \right)^{n_2} \left(\sum_{a=1}^{N} \phi_a^6 \right)^{n_3} \cdots . \tag{24}$$

We represent our fixed point by coordinates $Z_{\underline{n}}$. With each collection is associated a function

$$Z(\phi) = \sum_{\|\underline{n}\| \leq K} P_{\underline{n}}(\phi) \, Z_{\underline{n}}. \tag{25}$$

To define a suitable truncation, we introduce the norm $\|n\| := n_1 + 2n_2 + 3n_3 \dots$. In other words, we truncate the model to a maximal power of fields. The summation is restricted to the finite subset of $\mathbb{N}^\infty$ given by $\|n\| \leq K$.

Unfortunately, the functions (24) are not linearly independent. Moreover, the linear dependencies vary with N. Therefore, the representation (25) is not unique. As an illustration, the situation for $N = 2$ is studied in detail in the Appendix.

As we intend to use the over complete representation, we have to specify a lift of the renormalization group. We do this as follows. We have that

$$P_{\underline{n}}(\phi) \, P_{\underline{m}}(\phi) = P_{\underline{n}+\underline{m}}(\phi). \tag{26}$$

The Gaussian convolution can be written in the form

$$\int \mathrm{d}\mu_\gamma(\zeta) \, P_{\underline{m}}(\psi + \zeta) = \sum_{\|\underline{n}\| \leq K} P_{\underline{n}}(\psi) \, G_{\underline{n},\underline{m}}(N, \gamma), \tag{27}$$

where the coefficients are the following. We compute

$$\sum_{a=1}^{D} \frac{\partial^2}{\partial \phi_a^2} P_{\underline{m}}(\phi) = \sum_{\underline{n} \leq \underline{m}} P_{\underline{n}}(\phi) \, \mathcal{G}_{\underline{n},\underline{m}} \tag{28}$$

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with structure coefficients given by

$$\mathcal{G}_{\underline{n},\underline{m}} = 4ab\, m_a\, m_b \qquad \exists\, a \neq b: \begin{cases} m_a - 1 = n_a \\ m_b - 1 = n_b \\ m_{a+b-1} + 1 = n_{a+b-1} \\ \forall\, c \neq a, b: \ m_c = n_c \end{cases} \tag{29}$$

$$\mathcal{G}_{\underline{n},\underline{m}} = 4a^2\, m_a\, (m_a - 1) \qquad \exists\, a \neq 1: \begin{cases} m_a - 2 = n_a \\ m_{2a-1} + 1 = n_{2a-1} \\ \forall\, b \neq a: \ m_b = n_b \end{cases} \tag{30}$$

$$\mathcal{G}_{\underline{n},\underline{m}} = 2a\,(2a - 1)\, m_a \qquad \exists\, a \neq 1: \begin{cases} m_a - 1 = n_a \\ m_{a-1} + 1 = n_{a-1} \\ \forall\, b \neq a: \ m_b = n_b \end{cases} \tag{31}$$

$$\mathcal{G}_{\underline{n},\underline{m}} = 2\big(2(m_1 - 1) + N\big)m_1 \qquad \begin{cases} m_1 - 1 = n_1 \\ \forall\, a \neq 1: \ m_a = n_a \end{cases} \tag{32}$$

We then compute the Gaussian integral as the matrix exponential thereof. The result is

$$G_{\underline{n},\underline{m}}(N, \gamma) \;=\; \exp\left(\frac{\gamma}{2}\,\mathcal{G}\right)_{\underline{n},\underline{m}}. \tag{33}$$

The matrix $\mathcal{G}$ becomes upper triangular, when the couplings are sorted according to their total power of fields $\|\underline{n}\|$. (When the renormalization group is truncated to a finite power of fields, the Gaussian convolution alone does not generate higher powers.) Furthermore, the matrix $\mathcal{G}$ becomes nilpotent, and the matrix exponential becomes a finite sum. Our first system of equations is

$$R_K(Z)_{\underline{n}} \;=\; \beta^{2\,\|\underline{n}\|} \sum_{\|\underline{m}\| \leq K} G_{\underline{n},\underline{m}}(N, \gamma) \sum_{\underline{l}+\underline{k}=\underline{m}} Z_{\underline{l}}\, Z_{\underline{k}}\ . \tag{34}$$

This system of equations defines a lifting of the renormalization group. Every fixed point of (34) becomes through (25) a fixed point of the original hierarchical renormalization group. Their spectra of eigenvalues coincide (in the no-truncation limit), but some eigenvalues become degenerate. The eigenvalue problem is completely analogous to the $O(N)$-invariant case. We do not write down the equations here.

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K	#coups	K	#coups	K	#coups
1	2	10	139	20	2714
2	3	11	195	21	3506
2	4	12	272	22	4508
3	7	13	373	23	5763
4	12	14	508	24	7338
5	19	15	684	25	9296
6	30	16	915	26	11732
7	45	17	1212	27	14742
8	67	18	1597	28	18460
9	97	19	2087	29	23025

Table 5: Number of couplings in the lifted system for given truncation K.

We can use (34) to determine the cubic fixed point for any value of N. The reason is again that N enters polynomially in the structure coefficients (33). The drawback of this representation is that the number of couplings increases very fast with the order of truncation. Table (6.1) shows this number of couplings with $\|n\| \leq K$ for $K = 1 \ldots 29$.

We looked at the system up to $K = 18$. In this model one has to compute 1597 couplings. To get an idea of the achieved precision, we did one run with $K = 19$, which means 2087 couplings. (Has anyone ever computed a renormalization group with more unknowns?)

As in the $O(N)$ case we recognize one eigenvalue to become marginal at a critical number of components N_c. Again the value of N_c is suggested to be

$$2.2 \lesssim N_c \lesssim 2.25 \tag{35}$$

Table (7) combined with the investigation in polar coordinates for the $N = 2$-model, to be described below, suggests that the precision of μ_2 at $K = 18$ is three digits.

The calculation with $K = 19$ was only performed for the case $N = 2$ and $D = 3$, as it required about 22 hours CPU time on an IBM RS6000 workstation, $K = 18$ required around 10 hours on the same system.

As the value μ_2 decreases with increasing K, we expect the critical value N_c to be given an upper bound by the value $N_c < 2.25$.

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N	$\mu^{(1)}$	$\mu^{(2)}$	$\mu^{(3)}$	$\mu^{(4)}$	$\mu^{(5)}$	$\mu^{(5')}$
2.00	1.4337	1.0188	0.8540	0.5968	0.4887	0.4697
2.10	1.3988	1.0066	0.8551	0.5712	0.5367	0.4759
2.15	1.3857	1.0034	0.8540	0.5717	0.5545	0.4811
2.20	1.3790	1.0013	0.8531	0.5963	0.5500	0.4840
2.25	1.3765	0.9996	0.8525	0.6163	0.5481	0.4844
2.30	1.3758	0.9980	0.8521	0.6309	0.5480	0.4836
2.40	1.3761	0.9952	0.8515	0.6498	0.5491	0.4809
2.50	1.3766	0.9926	0.8508	0.6606	0.5498	0.4783
2.70	1.3765	0.9880	0.8490	0.6704	0.5489	0.4740
3.00	1.3735	0.9813	0.8451	0.6730	0.5442	0.4685

Table 6: Cubic spectrum at $K = 18$.

K	$\mu^{(1)}$	$\mu^{(2)}$	$\mu^{(3)}$
15	1.488	1.026	0.821
16	1.4557	1.0224	0.8399
18	1.43377	1.01889	0.85407
19	1.430341	1.018439	0.856580

Table 7: Cubic spectrum as function of K

To find a cubic fixed point by Newton iteration, one has start from a sufficiently close initial guess. We refrain from presenting a lengthy table of fixed point couplings. Instead, we present a sufficiently accurate initial guess from which anyone can reconstruct the cubic fixed point. (Later we will discuss the fixed point couplings in the special case when $N = 2$). Our search strategy was the following. We started at two ends of the interval of interest, at $N = 2$ and at $N = 3$. First we looked for the $O(N)$ fixed point, using the start values

$$Z_{(0,0,0,0,\dots)} = 0.4$$
$$Z_{(1,0,0,0,\dots)} = 0.06$$
$$Z_{(2,0,0,0,\dots)} = 0.001$$
$$Z_{(n,0,0,0,\dots)} = 0 \quad \forall n > 2.$$

Within our scheme (no rescaling on the $Z_{\underline{n}}$) this initial guess turned out to be sufficient to converge to the the $O(N)$ fixed point at both values of N. To find the cubic fixed point, we took the $O(N)$ fixed point and changed the following coefficients:

$$Z_{(0,0,0,0,\dots)} = 0.64$$
$$Z_{(0,1,0,0,\dots)} = 0.007$$
$$Z_{(0,0,1,0,\dots)} = 0.00011$$

for the case $N = 2$ and

$$Z_{(0,0,0,0,\dots)} = 0.45$$
$$Z_{(0,1,0,0,\dots)} = 0.007$$
$$Z_{(0,0,1,0,\dots)} = 0.00011$$

for $N = 3$. Having found the cubic fixed point at these two values of N we stepped towards the critical N_c from both sides, using the last fixed point as starting vector for the next iteration.

6.2 $N = 2$ Using Polar Coordinates

For fixed N, it seems natural to introduce spherical coordinates. For $N = 2$, we have that

$$Z(\phi) = Z(r, \varphi) = \sum_{m \geq 0} Z_m(r) \, \cos(m\varphi) \,. \tag{36}$$

The cubic symmetry can be implemented by requiring that the coefficients $Z_m(r)$ be zero for all m which are not integer multiples of 4. The result Z' of a hierarchical RG transformation applied to Z can be written in the form

$$Z'(\phi) = \mathrm{e}^{-\frac{1}{2}\beta^2 \phi^2} \int d^2\zeta \, \mathrm{e}^{-\frac{1}{2}\zeta^2} \, \mathrm{e}^{\beta \phi \zeta} \, Z^2(\zeta) \,. \tag{37}$$

We introduce polar coordinates for ζ and ϕ through

$$\zeta = r(\cos \varphi, \sin \varphi) \,, \quad \phi = R(\cos \theta, \sin \theta) \,, \tag{38}$$

and expand the square of Z

$$Z^2(\zeta) = \sum_{m \geq 0} \left(Z^2\right)_m (r) \, \cos(m\zeta) \,. \tag{39}$$

The $(Z^2)_m$ are related to the Z_m through

$$\left(Z^2\right)_m = \frac{1}{2} \sum_{m_1} \sum_{m_2} Z_{m_1} Z_{m_2} (\delta_{m,m_1+m_2} + \delta_{m,|m_1-m_2|}) \,. \tag{40}$$

The angle integration in eq. (37) can be performed, resulting in

$$Z'(R, \theta) = \sum_m Z'_m(R) \, \cos(m\theta) \,, \tag{41}$$

with

$$Z'_m(R) = \mathrm{e}^{-\frac{1}{2}\beta^2 R^2} \int_0^\infty dr \, r \, \mathrm{e}^{-\frac{1}{2}r^2} \, I_m(\beta r R) \, (Z^2)_m(r) \,. \tag{42}$$

Here, I_m denotes the modified Besselfunction of order m. The next step is an expansion of the Z_m in a power series of r^2,

$$Z_m = \sum_l Z_{ml} \, r^{2l} \,. \tag{43}$$

Expanding the Bessel function and performing the integration over r yields the following relation (for m even):

$$Z'_{ml'} = \sum_l C^m_{l'l} (Z^2)_{ml},$$ (44)

with

$$C^m_{l'l} = 2^{l-l'} \beta^{2l'} \sum_{j=0}^{l'-m/2} (-1)^j \frac{(l+l'-j)!}{(l'-\frac{m}{2}-j)!\,(l'+\frac{m}{2}-j)!\,j!}.$$ (45)

A combination of the transformation with the square operation eq. (40) yields the structure coefficients of the complete hierarchical RG transformation for $N = 2$. The matrix representing the linearized transformation is then easy to compute.

A Newton solver was used to find the cubic fixed point. It is known [1] that for $N = 2$ the cubical fixed point can be exactly mapped (by a rotation of the spin vector with an angle $\pi/4$) to the product of two independent Ising ($N = 1$) fixed points. Our fixed point, plotted in figure 4, is indeed of this type.

We determined the eigenvalues of the linearized transformation at that fixed point. Some care was devoted again to study the truncation effects. Table 8 shows the six leading eigenvalues at the cubical fixed points for three different truncation orders. To the given precision, it is sufficient to include $K = 32$ powers of r^2 in the ansatz. Furthermore, it is sufficient to include "angular momenta" m with $m \leq m_{\text{max}} = 4$. Going to $m_{\text{max}} = 5$ did not change the results for the exponents at all. From the factorization of the cubical fixed point into two Ising fixed points it follows that the spectrum can be built from the Ising spectrum through

$$\lambda_{i,\text{cub}} = \frac{1}{2} \lambda_{j,\text{Isi}} \lambda_{k,\text{Isi}}.$$ (46)

The leading Ising eigenvalue is 1.4271725 [13]. Its square divided by two is 1.0184107, in nice agreement with the result for the subleading eigenvalue of the cubic fixed point. The whole Ising spectrum is part of the cubic spectrum as follows when one of the Ising eigenvalues is taken equal to the volume eigenvalue two.

The comparison reveals that the spectrum in the polar coordinate representation and the spectrum in the overdetermined representation agree. Obviously, $K = 18$ does not bring about a very high degree of accuracy for the

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k	$K = 30$	$K = 26$	$K = 20$	$K = 19$	$K = 18$	μ
1	1.42717	1.42718	1.42867	1.43034	1.43377	1.4337
2	1.01841	1.01840	1.01831	1.01844	1.01889	1.0188
3	0.859411	0.859393	0.857925	0.856580	0.854073	0.8540
4	0.613255	0.613058	0.605915	0.602012	0.596873	0.5968
5	0.479634	0.479563	0.476318	0.473871	0.469780	0.4887
6	0.369252	0.368630	0.360018	0.357220	0.353718	0.4697

Table 8: The first 6 eigenvalues of the spectrum at the cubical fixed point, $N = 2$, for three different truncation orders. $m_{\max} = 4$ in all cases. For comparison the last column shows the results from the calculation with the over determined basis, truncation order $K = 18$, cf. table (6.1).

cubic invariant fixed point in the overdetermined representation. However, larger values of K would have required too much computer time. Furthermore, we learn that the overdetermined representation generates new spurious eigenvalues. They should converge to the true ones in the no-truncation limit. The first such spurious eigenvalue is however a subleading (or irrelevant) one.

It might be interesting to have a look at the fixed point couplings themselves. They are presented for $(K, m_{\max}) = (30, 4)$ in table 9.

7 Conclusions

In the framework of the hierarchical renormalization group, we have studied the stability of both the $O(N)$ symmetric and the cubical fixed point for $D = 3$, in the range between $N = 2$ and $N = 3$. No problems arised when investigating the stability of the $O(N)$ symmetric fixed point. For the cubical fixed point, however, the extension of a suitable basis to non-integer N turns out to be nontrivial. We solved this by using an over-complete set of functions. The big number of couplings that had to be used in this approach required quite an effort to solve the fixed point equations. Furthermore, it might be considered as a problem that the continuation from integer N to the real domain is by no means unique. The fact, however, that we find consistent values of N_c both at the $O(N)$ and the cubical fixed point seems to indicate that the chosen basis is a natural one. One further comment is in order: While in the full model the value of N_c is very close to 3, the

l	Z_{0l}	Z_{1l}	Z_{2l}	Z_{3l}	Z_{4l}
0	.5668E+00	.0000E+01	.0000E+01	.0000E+01	.0000E+01
1	.6404E–01	.0000E+01	.0000E+01	.0000E+01	.0000E+01
2	.3162E–02	.1518E–03	.0000E+01	.0000E+01	.0000E+01
3	.9355E–04	.1190E–04	.0000E+01	.0000E+01	.0000E+01
4	.1902E–05	.4309E–06	.3827E–08	.0000E+01	.0000E+01
5	.2877E–07	.9721E–08	.2351E–09	.0000E+01	.0000E+01
6	.3409E–09	.1550E–09	.6851E–11	.3149E–13	.0000E+01
7	.3280E–11	.1877E–11	.1269E–12	.1624E–14	.0000E+01
8	.2633E–13	.1807E–13	.1689E–14	.4026E–16	.1135E–18
9	.1799E–15	.1430E–15	.1728E–16	.6414E–18	.5103E–20
10	.1064E–17	.9518E–18	.1420E–18	.7407E–20	.1111E–21
11	.5515E–20	.5436E–20	.9673E–21	.6629E–22	.1567E–23
12	.2532E–22	.2703E–22	.5586E–23	.4798E–24	.1610E–25
13	.1039E–24	.1185E–24	.2783E–25	.2895E–26	.1290E–27
14	.3836E–27	.4624E–27	.1214E–27	.1488E–28	.8391E–30
15	.1283E–29	.1620E–29	.4688E–30	.6628E–31	.4568E–32
16	.3911E–32	.5135E–32	.1618E–32	.2593E–33	.2125E–34
17	.1090E–34	.1479E–34	.5024E–35	.9000E–36	.8583E–37
18	.2786E–37	.3890E–37	.1412E–37	.2791E–38	.3042E–39
19	.6533E–40	.9346E–40	.3596E–40	.7760E–41	.9517E–42
20	.1401E–42	.2046E–42	.8292E–43	.1934E–43	.2633E–44
21	.2728E–45	.4055E–45	.1719E–45	.4294E–46	.6415E–47
22	.4759E–48	.7182E–48	.3167E–48	.8407E–49	.1363E–49
23	.7311E–51	.1117E–50	.5100E–51	.1428E–51	.2490E–52
24	.9663E–54	.1493E–53	.7021E–54	.2061E–54	.3832E–55
25	.1068E–56	.1666E–56	.8042E–57	.2461E–57	.4847E–58
26	.9536E–60	.1499E–59	.7407E–60	.2353E–60	.4879E–61
27	.6560E–63	.1039E–62	.5239E–63	.1721E–63	.3738E–64
28	.3246E–66	.5174E–66	.2658E–66	.8996E–67	.2039E–67
29	.1023E–69	.1641E–69	.8570E–70	.2982E–70	.7026E–71
30	.1539E–73	.2479E–73	.1315E–73	.4694E–74	.1146E–74

Table 9: Coupling constants of the cubical fixed point, for $K = 30$ and $m_{\max} = 4$.

hierarchical $O(N)$ fixed point in three dimensions becomes unstable with respect to cubic perturbations already at $N_c = 2.219$. It has been observed in other contexts that the dependence of certain quantities on the number of spin components is shifted towards smaller values of N in the hierarchical approximation. For instance, the two dimensional hierarchical non-linear σ-model is asymptotically free for $N > 1$ as opposed to $N > 2$ in the full setting [5, 14]. This is due to the absence of wave function renormalization in the hierarchical approximation.

Supplementing the present work by a high order ε-expansion (which is certainly feasible) would be interesting and useful.

Acknowledgements

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Appendix: A Basis for the $N = 2$ Case

The polynomials (25) do not form a basis for cubic invariant functions. There are linear dependencies. The simplest relation among these is given by

$$P_{(0,2,0,0)} = 2P_{(0,0,0,1)} - 2P_{(2,1,0,0)} + P_{(4,0,0,0)}, \tag{47}$$

where the indices are to be understood in the sense of (24) with all n_i for $i > 4$ are equal to zero.

For the case $N = 2$ a basis for polynomials of type (25) is given by

$$P_{(n_o,n_c)}(\phi) := \left(\sum_{a=1}^{2} \phi_a^2 \right)^{n_o} \sum_{a=1}^{2} \phi^{4n_c}. \tag{48}$$

The relation with (24) is that (48) equals $P_{\underline{m}}$ with $m_1 = n_o$, $m_{4n_c} = 1$, and all others are zero.

This basis has the following complications. A product of two such polynomials is no more the polynomial to the index given by the sum of the indices of the factors. In general, a product of two basis elements is a linear combination

$$P_{(n_o,n_c)}(\phi)P_{(m_o,m_c)}(\phi) = \sum_{\substack{(k_o,k_c) \\ k_0=n_o+m_o \\ k_c=n_c+m_c}} C_{(k_o,k_c)}^{(n_o,n_c),(m_o,m_c)} P_{(k_o,k_c)}(\phi). \tag{49}$$

We do not have closed expressions for the coefficients $C_{(k_o,k_c)}^{(n_o,n_c),(m_o,m_c)}$. We tabulated those we needed in our programs. The table was generated using computer algebra.

The Gaussian integration of polynomials of type (48) is given by

$$\int d\mu_\gamma(\zeta) \, P_{(n_o,n_c)}(\psi + \zeta) = \sum_{k_o=0}^{n_o} \sum_{k_c=0}^{n_c} I_{k_o,k_c}^{n_o,n_c}(N,\gamma)P_{(k_o,k_c)}(\psi) + \\ \sum_{k_o=0}^{n_o} \sum_{k_c=0}^{n_c} I_{k_o,k_c+\frac{1}{2}}^{n_o,n_c}(N,\gamma)P_{(k_o,k_c+\frac{1}{2})}(\psi) \tag{50}$$

with

$$I_{k_o,k_c}^{n_o,n_c}(N,\gamma) = G_{2n_c,2k_c}(1,\gamma)G_{n_o,k_o}(4n_c + 4k_c + N,\gamma), \tag{51}$$

where $G_{n,m}(N, \gamma)$ is as in (9). The formula (50) does not yet provide a formula for the Gaussian integration in the basis (48), as the second term is not yet expanded in this basis. Its expansion involves again rather complicated structure coefficients which we determined by means of computer algebra.

We used this representation to check our results with the overdetermined representation. This polynomial basis turned out to a less appropriate representation than the following one with polar coordinates. Both suffer the drawback that they apply to $N = 2$ only, and cannot be continued to any N. (They can be continued but do not coincide with the desired models at integer N.)
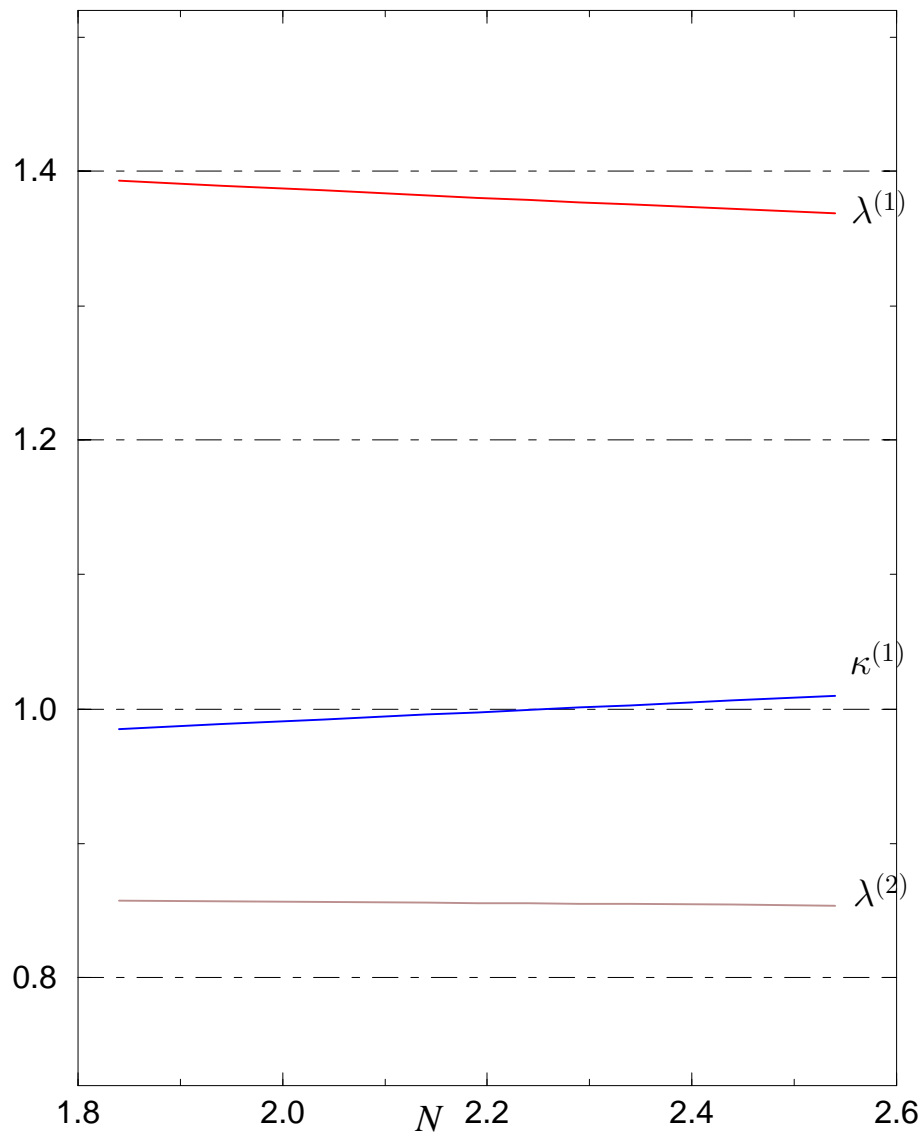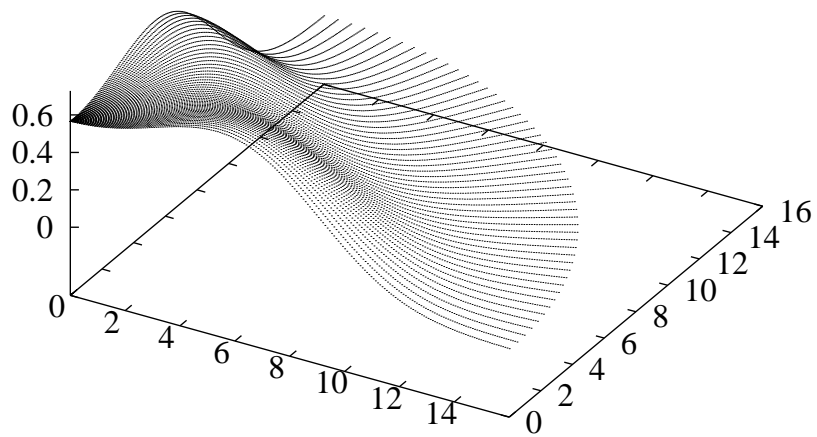
Figure 3: Visualization of Table (3)

Figure 4: Cubical fixed point Boltzmann factor Z^* for $N = 2$, as function of the two spin components ϕ_1 and ϕ_2.